Relativistic analogue of the D'Alembert principle of virtual displacements and the effective actions for the spherically-symmetric charged dust shell in General Relativity

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Abstract

A simple and direct derivation of the variational principle and effective actions for a spherical charged dust shell in general relativity is offered. This principle is based on the relativistic version of the D'Alembert principle of virtual displacements and leads to the effective actions for the shell, which describe the shell from the point of view of the exterior or interior stationary observers. Herewith, sides of the shell are considered independently, in the coordinates of the interior or exterior region of the shell. Canonical variables for a charged dust shell are built. It is shown that the conditions of isometry of the sides of the shell lead to the Hamiltonian constraint on these interior and exterior dynamical systems. Special cases of the «hollow» and «screening» shells are briefly considered, as well as a family of the concentric charged dust shells.

1 Introduction

The theory of spherically-symmetric thin shells plays key role for construction effective non-trivial models for the collapsing gravitating configurations. Really, thin shells have been found to have widespread application in different regions of the General Relativity, astrophysics and cosmology for modeling of the extended objects which thickness can be neglected. For example, they are intensively used for the analysis of the basic problems of a gravitational collapse, including its classical and quantum aspects. In astrophysics, extending spherical shells are used for modeling of the supernovas and changing space objects. At larger scales, specific configurations of shells have also been considered to construct cosmological models, to analyze phase transitions in the early universe or to describe cosmological voids etc. (see for reviews [1]).

The equations of motion of the spherically-symmetric shells have been obtained in [2], the equations of motion of the charged spherical shells have been found in [3, 4, 5]. The construction of the variational principle, as well as Lagrangian and canonical formalism for these objects were discussed, for example, in [6]-[11]. There are a number of problems here, the most basic of which is the dependence on the choice of the evolution parameter (internal, external, proper). It turns out that the selection of the time coordinate affects the choice of the quantization scheme, leading, in general, to the unitarily non-equivalent quantum theories.

In the majority of the works the variational principle for shells is constructed in the co-moving frame of reference, or in one of the variants of the freely falling frames of reference. In our opinion, the choice of the exterior or interior remote stationary observer in the theory of the gravitating shells is the most natural and corresponds with the real physics. The natural Hamiltonian formalism of the neutral self-gravitating shell was considered in [11, 12], where the Hamiltonian of the shell is actually postulated.

The general approach to the construction of a variational principle for the spherically-symmetric dust shells in terms of the stationary interior and exterior observers was developed on the basic principles in [13, 14, 15]. This approach is based on a cumbersome procedure for reduction of the full action, which contains the Einstein-Hilbert terms for the inner and outer regions, the action for the dust matter on the singular shell, the surface matching and normalizing terms, the surface terms similar to Gibbons-Hawking boundary surface terms [16], which are introduced to fix the metric on the boundary of the considered region, and on the subsequent modification of the variational procedure.

In this paper a simple and direct construction of the variational principle for the charged spherically-symmetric dust shells in general relativity is proposed. This procedure is based on the generalization of the relativistic version of the D'Alembert principle of virtual displacements [18], which has been suggested in [17] for

the construction of the effective actions for the neutral dust shell. The effective actions for the charged spherical shell in the coordinates of the inner and outer regions of space-time are constructed and the Hamiltonian constraint is obtained, which plays the role of the integrals of motion.

Everywhere in this paper we suppose that the gravitational constant $k = 1$ and the speed of light $c = 1$. The metric tensor $g_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3)$ has signature (+ - - -).

2 Spherically-symmetric space-time with spherical shell

Let us consider a spherically-symmetric configuration $D = D_- \cup \Sigma \cup D_+$ which is the union of the concentric interior D_- and exterior D_+ regions. These regions are matched together along time-like spatially closed hypersurface Σ which forms world sheet of the spherical infinitely thin dust shell with dust density σ and charge density σ_e. Let $x^i : \{x^2 = \theta,\ x^3 = \alpha\}$ $(i, k = 2, 3)$ be the general angular, and $x^a_\pm (a, b = 0, 1)$ be the individual coordinates defined in regions $D_\pm$. Then gravitational fields in regions $D_\pm$ are generally described by metrics

$$^{(4)}ds^2_\pm = {}^{(4)}g_{\mu\nu} dx^\mu dx^\nu = {}^{(2)}ds^2_\pm - r^2 d\sigma^2, \tag{1}$$

$$^{(2)}ds^2_\pm = \gamma^\pm_{ab} dx^a_\pm dx^b_\pm \,, \qquad d\sigma^2 = h_{ij} dx^i dx^j = d\theta^2 + \sin^2\theta d\alpha^2. \tag{2}$$

The two-dimensional metrics $\gamma^\pm_{ab}$ and the scale factor r are functions of the coordinates $x^a_\pm$. We have $^{(2)}ds_+|_\Sigma =^{(2)} ds_+|_\Sigma =^{(2)} ds_+$, as well as the world line of the shell is set by the equation $x^a = x^a(s)$. In the points on hypersurface we will define orthonormal basis

$$\{u^a,\ n^a\}\,, (u_a u^a = -n_a n^a = 1, \quad u_a n^a = 0)\,, \tag{3}$$

here u^a is a tangent vector of the shell's world line so $u^a_\pm|_\Sigma = dx^a/{}^{(2)}ds$, and n^a is the normal vector to the Σ which is directed from the region D_- to the D_+. From the formula (3) we find the equalities

$$n_0 = \sqrt{-g} u^1, \qquad n_1 = -\sqrt{-g} u^0, \tag{4}$$

$$u_0 = \sqrt{-g} n^1, \qquad u_1 = -\sqrt{-g} n^0. \tag{5}$$

We used $\gamma = \det|\gamma_{ab}|$.

3 Equations of motion of the spherically-symmetric charged dust shell

In the vacuum, the spherically-symmetric gravitational field of the charged source is described by the Reissner-Nordström metrics. In curvature coordinates, we can write the metrics for the regions D_- and D_+ as follows

$$^{(4)}ds^2_\pm = F_\pm dt^2_\pm - F^{-1}_\pm dr^2 - r^2 \left(d\Theta^2 - \sin^2\Theta d\alpha^2\right), \tag{6}$$

where

$$F_+ = 1 - \frac{2kM_+}{r} + \frac{kQ^2_+}{r^2}, \qquad F_- = 1 - \frac{2kM_-}{r} + \frac{kQ^2_-}{r^2}. \tag{7}$$

Here t_+ and t_- are the Keeling time coordinates in the exterior D_+ and interior D_- regions, accordingly; M_+ and M_- are the active masses; Q_+ and Q_- are the electric charges. These charges also generate electric fields with potintials $\varphi_\pm = Q_\pm/r$ in regions $D_\pm$.

For the spherically-symmetric charged dust shell the motion equations have the form (e.g. see [4])

$$n_a \frac{Du^a}{ds}|_+ + n_a \frac{Du^a}{ds}|_- = \frac{2}{\sigma}\left[T_{\alpha\beta} n^\alpha n^\beta\right], \tag{8}$$

$$n_a \frac{Du^a}{ds}|_+ - n_a \frac{Du^a}{ds}|_- = 4\pi\kappa\sigma = \frac{\kappa m}{r^2}, \tag{9}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor, $m = 4\pi\sigma r^2$ is the rest mass of the shell and $Du^a = u^a_{;b} dx^b$ indicates covariant differential relative to the metrics γ_{ab}. The symbol $[\Phi] = \Phi|_+ - \Phi|_-$ denotes the jump of the quantity Φ on Σ. The signs $|_+$ or $|_-$ indicate the marked quantities to be calculated as the limit values when approaching the boundary Σ from inside and outside, respectively. In our case (see [4]):

$$\left[T_{\alpha\beta} n^\alpha n^\beta\right] = \frac{q}{8\pi r^4}\left(Q_+ + Q_-\right), \qquad q = Q_+ - Q_-, \tag{10}$$

where q is the charge of the shell.

Two-dimensional equations of motion (8) and (9) allow us to obtain the independent equations of motion of the shell in the coordinates for each of the two-dimensional areas $D_{+}^{(2)}$ and $D_{-}^{(2)}$ separately. Taking into account the relations (4), (5) and the equations (8), (9) we obtain the equations of motion of the shell in terms of quantities with respect only to the $D_{+}^{(2)}$, or only to the $D_{-}^{(2)}$

$$u^a_{;b}u^b|_{\pm} = \frac{Du^a}{ds}|_{\pm} = \frac{1}{m}\left(\frac{qQ_{\pm}}{r^2} \pm \frac{\kappa m^2 - q^2}{2r^2}\right) n^a. \tag{11}$$

We see that the regions $D_{+}^{(2)}$ and $D_{-}^{(2)}$, and their boundaries $\Sigma_{\pm}^{(1)}$ together with the corresponding gravitational and electric fields can be considered separately and independently, as the manifolds $D_{\pm}^{(2)}$ with the edges $\Sigma_{\pm}^{(1)}$.

4 The variational principle for a spherically-symmetric charged dust shell

Following [17], we rewrite the equations of motion for a shell (11) in the form, which is similar to the equations of motion for a charge in an electromagnetic field

$$u^a_{;b}u^b|_{\pm} = -G^{\pm}_{ab}u^b|_{\pm}, \tag{12}$$

where

$$G^{\pm}_{ab} = -G^{\pm}_{ba}, \qquad G^{\pm}_{01} = \frac{1}{m}\left(-\frac{qQ_{\pm}}{r^2} \pm \frac{q^2 - \kappa m^2}{2r^2}\right). \tag{13}$$

To obtain this result the relation (10) has been used with the fact that for the Reissner-Nordström metrics we have $\sqrt{-\gamma} = 1$. The motion equations (12) define the trajectories $x^a_{\pm} = x^a_{\pm}(s)$ corresponding to the real motions of a shell in the regions $D_{\pm}^{(2)}$.

Now we will consider other possible trajectories $\tilde{x}^a_{\pm}(s) = x^a_{\pm}(s) + \delta x^a_{\pm}(s)$, which are close enough to the real trajectory. From the relation (12) it follows that

$$\left(\left(u^a_{;b}u^b + G_{ab}u^b\right)\delta x^a\right)_{\pm} = 0. \tag{14}$$

This equation is a relativistic analogue of the D'Alambert principle of virtual displacement when the time coordinate $x^0 = x^0(s)$ and the spatial one $x^1 = x^1(s)$ are considered as the dynamic variables. Let us multiply expressions (14) by ${}^{(2)}ds$ and integrate the result along a trajectory γ, then we get

$$\int\limits_{\gamma}\left(\left(u^a_{;b}u^b + G_{ab}u^b\right)\delta x^a\right)^{(2)}_{\pm} ds = 0. \tag{15}$$

First term in this formula can be transformed using the variational relation

$$\delta|^{(2)}_{\pm} ds|_{\pm} = -\left(u^a_{;b}u^b\delta x^{a(2)}ds\right)|_{\pm} + d\left(u_a\delta x^a\right)|_{\pm}. \tag{16}$$

Second term in the formula (15) can be written as follows

$$\left(G_{ab}u^b\delta x^a\right)^{(2)}_{\pm} ds = \left(G_{01}\left(dx^0\delta x^1 - dx^1\delta x^0\right)\right)_{\pm}. \tag{17}$$

In the regions $D_{\pm}^{(2)}$, let us introduce the auxiliary continuous and invariant one-forms $\beta^{\pm} = B^{\pm}_a x^a$ by means of the relation

$$d\beta^{\pm} = G^{\pm}_{01}\left(dx^0 \wedge dx^1\right)_{\pm}. \tag{18}$$

Here $B^{\pm}_a = B^{\pm}_a\left(x^0, x^1\right)$ is a vector potential of the gravitational and Coulomb self-actions, and $G^{\pm}_{ab} \equiv B^{\pm}_{b,a} - B^{\pm}_{a,b}$ is its intensity in the exterior and interior regions $D_{\pm}^{(2)}$, accordingly. Note that in two-dimensional space the integrability condition for the relation (18) holds identically. Making use of the definition (18) and the formula

$$d\left(B_a\delta x^a\right)_{\pm} - \delta\left(B_a\delta x^a\right)_{\pm} = G^{\pm}_{01}\left(dx^0\delta x^1 - dx^1\delta x^0\right)_{\pm}, \tag{19}$$

we obtain second term in (15) as

$$\left(G_{ab}u^b\delta x^a\right)|_{\pm}^{(2)}ds=\left(d\left(B_a\delta x^a\right)-\delta\left(B_a dx^a\right)\right)|_{\pm}. \tag{20}$$

Substituting the expressions (16) and (20) to the equation (15), we find the sought-for variational formulae

$$\delta\int\limits_{\gamma}\left({}^{(2)}ds-B_a dx^a\right)_{\pm}+\left\{\left(u_a+B_a\right)\delta x^a\right\}_{\pm}|_A^B=0, \tag{21}$$

where indices A and B indicate that corresponding quantities are taken in the initial and final position of the shell. Hence, it follows that for all real trajectories $x_{\pm}^a=x_{\pm}^a(s)$ the integrals

$$I_{sh}^{\pm}=-m\int\limits_{\gamma}\left({}^{(2)}ds-B_a dx^a\right)_{\pm}=-m\int\limits_{\gamma}\left({}^{(2)}ds-\beta\right)_{\pm}, \tag{22}$$

have stationary values $\left(\delta I_{sh}^{\pm}=0\right)$ with respect to the arbitrary possible variations of the shell motions, when initial and final positions remain fixed, i.e. $\left(\delta x_{\pm}^a\right)|_A^B=0$. Thus, the requirement of stationarity $\delta I_{sh}^{+}=0$ and $\delta I_{sh}^{-}=0$ with respect to the arbitrary variations of the coordinates δx^+ and δx^- yield to the equations of motion of a charged dust shell (12) in the coordinates x^+ and x^-, respectively. Hence, it can be seen that I_{sh}^{-} is the effective action for the charged dust shell, defined only in the coordinates x_{-}^a (i.e. in the interior region), and I_{sh}^{+} is the effective action of the shell, defined only in coordinates x_{+}^a in the exterior region.

The proposed procedure for the construction of the effective action is the relativistic analogue of the classic method of deriving the integral Hamilton principle from the D'Alembert principle of virtual displacements [18].

5 The effective actions and Lagrangians for the spherical charged dust shell in curvature coordinates

The effective actions (22) for the spherical charged dust shell are obtained in the invariant form. Now, we will find effective actions of this shell in the gravitaional Reissner-Nordström field (6). Using curvature coordinates, we choose common spatial spherical coordinates $\{r,\theta,\alpha\}$ in $D_{\pm}$, and individual time coordinates $t_{\pm}$ in $D_{\pm}$, respectively. Then the world sheet of the shell Σ is given by equations $r=r_-\left(t_-\right)$ and $r=r_+\left(t_+\right)$, respectively.

In this case, from definition (18) we obtain

$$d\beta^{\pm}=\left(G_{01}^{\pm}\left(r\right)dt\wedge dr\right)_{\pm}=-dt_{\pm}\wedge dV^{\pm}=d\wedge\left(V^{\pm}dt_{\pm}\right), \tag{23}$$

where

$$mV^{\pm}=-q\varphi_{\pm}\pm U, \tag{24}$$

and

$$U=\frac{q^2-km^2}{2r} \tag{25}$$

is the full effective potential energy of the gravitational and electromagnetic self-actions of the shell. First term in 24 is the interaction energy of the shell's charge q and electric fields with potential $\varphi_{\pm}=Q_{\pm}/r$ in regions $D_{\pm}^{(2)}$, respectively. The general solutions of the equations in exterior derivatives (23) for each of two regions $D_{\pm}^{(2)}$ can be written in the forms

$$\beta^{\pm}=V^{\pm}dt_{\pm}+d\psi_{\pm}=\frac{1}{m}\left(-q\varphi_{\pm}\pm U\right)dt_{\pm}+d\psi_{\pm}, \tag{26}$$

where $\psi_{\pm}=\psi_{\pm}\left(t_{\pm},r\right)$ is a function which sets calibration of the vector potential B_a in the regions $D_{\pm}^{(2)}$. These functions can be chosen so that one-form β will be continuous on the shell (namely $\beta^+|_{\Sigma}=\beta^-|_{\Sigma}=\beta$).

Substituting one-form (26) to the actions (22), we get the general representation of the effective actions for the charged dust spherical shell in the form

$$I_{sh}^{\pm}=-\int\limits_{\gamma}\left\{m^{(2)}ds+\left(q\varphi_{\pm}\mp U\right)dt-md\psi\right\}|_{\pm}. \tag{27}$$

Making use of the gauge conditions $\psi_\pm = 0$ in each of the regions $D^{(2)}_\pm$ the effective actions for the charged dust spherical shell can be written as

$$I^\pm_{sh} = \int\limits_\gamma L^\pm_{sh} dt|_\pm = I^\pm_{sh} = -\int\limits_\gamma \left\{ m^{(2)} ds + (q\varphi_\pm \mp U)\, dt \right\} |_\pm . \tag{28}$$

Here the effective Lagrangians have been introduced in the form

$$L^\pm_{sh} = -m \left(\frac{^{(2)}ds}{dt} \right)_\pm - q\varphi_\pm \pm U = -m\sqrt{F_\pm - F_\pm^{-1} r_{t\pm}^2} - q\varphi_\pm \pm U. \tag{29}$$

They describe dynamics of a charged dust spherical shell from the point of view of the interior or exterior stationary observers. Here, for the sake of simplification, the radial velocity is denoted by $r_{t\pm} = dr/dt_\pm$. Note that one-form $\beta^\pm = \varphi^\pm (t_\pm, r)\, dt_\pm$ is not continuous on the shell Σ any more.

In the limiting case of small m and q, it can be formally put $M_+ = M_- = M$, $Q_+ = Q_- = Q$ and $U = 0$. Then the Lagrangians (29) will describe the test charged shell with mass m and charge q, which moves in the gravitational Reissner-Nordström field with parameters M and Q, and in the electric field with potential $\varphi = Q/r$.

6 The isometry condition and the Hamiltonian constraint

The effective actions $I^\pm_{sh}$ independently determine dynamics of the shell in the regions $D^{(2)}_\pm$. Therefore, the regions $D^{(2)}_\pm$ together with the boundaries $\Sigma^{(1)}_\pm$ and the corresponding fields can be considered separately and independently. The boundaries $\Sigma^{(1)}_\pm$ acquire the physical sense of the different faces of a dust shell with world sheet Σ^1 if regions $D^{(2)}_\pm$ are joined along these boundaries. However, this requirement can be realized only if the condition of the isometry for the boundaries $\Sigma^{(1)}_\pm$

$$F_+ dt_+^2 - F_+^{-1} dr^2 = F_- dt_-^2 - F_-^{-1} dr^2 = d\tau^2, \tag{30}$$

is fulfilled. Here τ is the intrinsic time of the shell. Thus we obtain $\Sigma^{(1)}_+ = \Sigma^{(1)}_- = \Sigma^{(1)}$, $\gamma_+ (t_+) = \gamma_- (t_-) = \gamma$.

It can be shown easily that the condition of isometry of the boundaries (30) leads to the Hamiltonian constraints. First of all, we have the relationships for the velocities

$$\frac{F_+}{r_{t+}} - \frac{1}{F_+} = \frac{F_-}{r_{t-}} - \frac{1}{F_-}, \tag{31}$$

$$r_\tau^2 \equiv \left(\frac{dr}{d\tau} \right)^2 = \frac{r_{t\pm}^2}{F_\pm - F_\pm^{-1} r_{t\pm}^2}, \qquad r_{t\pm}^2 \equiv \left(\frac{dr}{dt_\pm} \right)^2 = \frac{F_\pm^2 r_\tau^2}{F_\pm + r_\tau^2}. \tag{32}$$

Further, from the Lagrangians (29) we find the momenta and Hamiltonians for the shell

$$P_\pm = \frac{\partial L^\pm_{sh}}{\partial r_{t\pm}} = \frac{m r_{t\pm}}{F_\pm \sqrt{F_\pm - F_\pm^{-1} r_{t\pm}^2}} = \frac{m}{F_\pm} r_\tau, \tag{33}$$

$$H^\pm_{sh} = \sqrt{F_\pm \left(m^2 + F_\pm P_\pm^2 \right)} + q\varphi_\pm \mp U = m\sqrt{F_\pm + r_\tau^2} + q\varphi_\pm \mp U = E_\pm . \tag{34}$$

Here $E_\pm$ are the energies, which are conjugated to the coordinate times $t_\pm$ and are conserved in the frames of reference of the respective stationary observers (interior or exterior one). Eliminating the velocity r_τ from (33) and (34) , the condition of isometry for the boundarieds $\Sigma^{(1)}_\pm$can be written as

$$F_+ P_+ = F_- P_-, \tag{35}$$

$$(E_+ - q\varphi_+ + U)^2 - m^2 F_+ = (E_- - q\varphi_- - U)^2 - m^2 F_- . \tag{36}$$

Substituting $\varphi_\pm = Q_\pm / r$ into the last equation and making use of the equations (7), (25) we obtain

$$H^+_{sh} = H^-_{sh} = M_+ - M_- = E. \tag{37}$$

Here $E = E_+ = E_-$ denotes the total energy of the shell, which is conjugated both to the coordinate times t_+ and t_-, and its value is independent of the stationary observer's position (inside or outside of the shell).

Thus, the dynamic system described by the Lagrangians $L^{\pm}_{sh}$ are not independent. They satisfy the Hamiltonian constraint (37), which ensure the isometry of the shell faces.

Hamiltonian constraint (37), using the relation (34), can be rewritten in terms of momenta in square form

$$F_{\pm}^{-1}\left(M_+ - M_- - q\varphi_{\pm} \pm U\right)^2 - F_{\pm}P_{\pm}^2 = m^2. \tag{38}$$

Hence, taking into account $P_{\pm} = -dS_{\pm}/dr$ we find the stationary Hamilton-Jacobi equation

$$F_{\pm}^{-1}\left(M_+ - M_- - q\varphi_{\pm} \pm U\right)^2 - F_{\pm}\left(\frac{dS_{\pm}}{dr}\right)^2 = m^2, \tag{39}$$

where S is the reduced action.

Now, we derive the first-order differential equations of motion for the charged dust shells. For this purpose we rewrite the Hamiltonian constraint (37), using the formulae (34) and (25), in the form

$$m\sqrt{F_{\pm} + r_{\tau}^2} = [M] - \frac{qQ_{\pm}}{r} \pm \frac{q^2 - km^2}{2r}. \tag{40}$$

These equations are usually written in mixed form

$$m\sqrt{F_- + r_{\tau}^2} + m\sqrt{F_+ + r_{\tau}^2} = 2\left(M_+ - M_-\right) - \frac{q\left(Q_- + Q_+\right)}{r}, \tag{41}$$

$$m\sqrt{F_- + r_{\tau}^2} - m\sqrt{F_+ + r_{\tau}^2} = \frac{\kappa m^2}{r}. \tag{42}$$

Note that this formulas are reasonable outside the event horizon, where the curvature coordinates are valid. Formally, we can use these formulas under the horizon too, i.e. in T^-- and T^+-regions, assuming r to be the time coordinate. It turns out that in order to use the simplicity and convenience of the curvature coordinates and to conserve the information about the shells in the region R^-, it is sufficiently to introduce an additional discrete variable $\epsilon = \pm 1$ and perform the replacement ${}^{(2)}ds_{\pm} \to \epsilon_{\pm}^{(2)}ds_{\pm}$ in the actions $I^{\pm}_{sh}$ (28) (fore more on the neutral shells, see [13, 14]). Here, $\epsilon_{\pm} = 1$ corresponds to the region R^+, and $\epsilon_{\pm} = -1$ – to the region R^-. Then, for the extended system, Hamiltonians (34) take the form

$$H^{\pm}_{sh} = \epsilon_{\pm}\sqrt{F_{\pm}\left(m^2 + F_{\pm}P_{\pm}^2\right)} + q\varphi_{\pm} \mp U \tag{43}$$

7 Special cases of dust shells

7.1 Hollow and screening shells

Because isometry of the sides of the shell and the Hamiltonian constraint, the considerations of the shell in terms of interior or exterior frames of reference are equivalent. Therefore, we can choose the coordinates in which the equations of motion have the most simple and most convenient form. For example, the equations of motion of charged shells are greatly simplified when one of the regions of space-time, inside or outside of the shell, is flat. Thus, for "hollow" shell the coordinates of the inner region are convenient, and for the "screening" shell so the coordinates of the outer region are.

In the first case we have a self-gravitating shell, for which $M_- = 0$ and $Q_- = 0$. Such a shell, in the interior coordinates of flat space-time, moves only under the influence of the potential energy U of the gravitational and the electric self-interactions (25), which depends only on its rest mass m and charge q. Let us use the following notations $M_+ = M$ and $Q_+ = Q$. Then the exterior region D_+ of the shell is described by Reissner-Nordström metrics (6), where $F_+ = F = 1 - 2kM/r + kQ^2/r^2$. In terms of the coordinates $\{t_+, r\}$, the Lagrangian, the Hamiltonian and the Hamiltonian constraint can be written as

$$L^{+}_{sh} = -m\sqrt{F - F^{-1}r_{t_+}^2} - \frac{qQ}{r} + \frac{q^2 - km^2}{2r}\,, \tag{44}$$

$$H^{+}_{sh} = \sqrt{F\left(m^2 + FP_+^2\right)} + \frac{qQ}{r} - \frac{q^2 - km^2}{2r} = M\,, \qquad P_+ = \frac{mR_{t+}}{F\sqrt{F - F^{-1}R_{t+}^2}}\,, \tag{45}$$

$$F^{-1}\left(M - q\varphi_+ + \frac{q^2 - km^2}{2r}\right)^2 - FP_+^2 = m^2 \,. \tag{46}$$

In the interior region D_- we have $F_- = 1$. In terms of the coordinates $\{t_-, r\}$, the Lagrangian, the Hamiltonian and the Hamiltonian constraint are much simpler and they have the form

$$L_{sh}^- = -m\sqrt{1 - r_{t-}^2} - \frac{q^2 - km^2}{2r}, \tag{47}$$

$$H_{sh}^- = \sqrt{m^2 + P_-^2} + \frac{q^2 - km^2}{2r} = M\,, \qquad P_- = \frac{mr_{t-}}{\sqrt{1 - r_{t-}^2}}\,, \tag{48}$$

$$\left(M - \frac{q^2 - km^2}{2r}\right)^2 - P_-^2 = m^2 \,. \tag{49}$$

In the case of the «screening» shell $M_+ = 0$ and $Q_+ = 0$, and we put $M_- = -M$, $Q_- = Q$. Thus, the system has a nontrivial electric and gravitational fields only in the interior region D_- of the shell. The easiest way is to describe such a shell in terms of the coordinates $\{t_+, r\}$ of the exterior region, where it is moving under the influence of the potential energy of the gravitational and the electric self-interaction of the same form as for a «hollow» shell, but with the opposite sign. Thus, we have the Lagrangian, Hamiltonian and Hamiltonian constraint in the form

$$L_{sh}^+ = -m\sqrt{1 - r_{t+}^2} + \frac{q^2 - km^2}{2r}\,, \tag{50}$$

$$H_{sh}^+ = \sqrt{m^2 + P_+^2} - \frac{q^2 - km^2}{2r} = M\,, \qquad P_+ = \frac{mr_{t+}}{\sqrt{1 - r_{t+}^2}}\,. \tag{51}$$

$$\left(M + \frac{q^2 - km^2}{2r}\right)^2 - P_+^2 = m^2\,, \tag{52}$$

correspondingly.

7.2 A family of concentric charged dust shells

Let us briefly consider a more complex configuration, consisting of a set of concentric charged dust shells. Let R_a, m_a, q_a, τ_a be the radius, the proper mass, the charge, and the proper time of the a-th shell, respectively ($a = 1, 2, ..., N$). We assume that $R_b > R_a$ if $b > a$. Suppose that M_a, Q_a are the active mass and the electric charge that form the gravitationa; Reissner-Nordström field $F_a = 1 - 2kM_a/r + kQ_a^2/r^2$ in the area $R_a < r < R_{a+1}$, between a-th and $(a+1)$-th shells. We denote by F_a^-, φ_a^- and F_a^+, φ_a^+ the metric coefficients and the electric potentials in the neighborhood of a-th shell, in its interior $R_{a-1} < r < R_a$ and exterior $R_a < r < R_{a+1}$ regions, respectively. Then

$$F_a^- = 1 - \frac{2kM_{a-1}}{r} + \frac{kQ_{a-1}^2}{r^2}, \qquad F_a^+ = F_a = 1 - \frac{2kM_a}{r} + \frac{kQ_a^2}{r^2}\,, \tag{53}$$

$$\varphi_a^- = \frac{Q_{a-1}}{r}\,, \qquad \varphi_a^+ = \frac{Q_a}{r}\,, \qquad U_a = \frac{q_a^2 - km_a^2}{2r}\,, \tag{54}$$

where U_a is the potential energy of self-interaction of the a-th shell. Note that $F_a^+ = F_{a+1}^-$, $\varphi_a^+ = \varphi_{a+1}^-$ and $q_a = Q_a - Q_{a-1}$. In this case

$$H_a^\pm = \epsilon_a^\pm\sqrt{F_a^\pm\left(m^2 + F_a^\pm\left(P_a^\pm\right)^2\right)} + q\varphi_a^\pm \mp U_a \tag{55}$$

are the Hamiltonians of the a-th shell, which, as well as the momenta of the shells

$$P_a^\pm = \frac{m_a dr_a}{F_a^\pm d\tau_a}\,, \tag{56}$$

are considered relatively to the coordinates of areas $R_{a-1} < r < R_a$ and $R_a < r < R_{a+1}$, respectively. Here $\epsilon_a^{\pm} = \pm 1$. These Hamiltonians satisfy the constraints

$$H_a^+ = H_a^- = M_a - M_{a-1}. \tag{57}$$

The total Hamiltonian of the configuration

$$H = \sum_{a=1}^{N} H_a^{\pm} = \sum_{a=1}^{N} \left\{ \epsilon_a^{\pm} \sqrt{F_a^{\pm} \left(m^2 + F_a^{\pm} \left(P_a^{\pm} \right)^2 \right)} + q\varphi_a^{\pm} \mp U_a \right\}, \tag{58}$$

by virtue of the Hamiltonian constraints (57) is provided to be equal

$$H = E_{tot} = M_N - M_0. \tag{59}$$

Full electric charge of the configuration, because of the additivity of the charge, is

$$Q = \sum_{a=1}^{N} q_a = Q_N - Q_0 \tag{60}$$

If $M_0 = 0$ and Q_0, the system is moving in its own gravitational and electric fields. In this case $H_1^{\pm} = M_1$, $Q_1 = q_1$ and the full Hamiltonian and the charge of the system are

$$H = M_N = M, \qquad Q = Q_N \tag{61}$$

where $M = M_N$ is the full active mass of the configuration.

8 Conclusions

A special feature of the dynamics of the spherical shell is that its evolution is not accompanied by radiation and can be reduced to a simple Lagrangian system. This dynamic system has only one local degree of freedom $r = r(\tau)$. Therefore, there is a possibility of constructing equations of motion of the shell in terms of coordinates assigned only to the interior or exterior region and their independent consideration. Hence, making use of the simple generalization of the relativistic version of the D'Alambert principle of virtual displacements, the effective actions $I_{sh}^{\pm}$ (28) are constructed for a charged dust shell, describing its dynamics from the point of view of the exterior or interior stationary observers. This leads to the different effective Lagrangians $L_{sh}^{\pm}$ (29) and Hamiltonians $H_{sh}^{\pm}$ (34) of the shell in the interior and exterior regions $D_{\pm}^2$ with coordinates $x_{\pm}^a$. It turns out that the dynamical systems described by these Lagrangians are not independent. They satisfy the Hamiltonian constraint $H_{sh}^+ = H_{sh}^- = M_+ - M_- = E$, which guarantees isometry of the sides of the shell. The total energy of the shell $E = M_+ - M_-$ conjugates both time coordinates $t_{\pm}$ in the regions D, and its value is constant and it does not depend on the position of a resting observer inside or outside of the shell.

Consideration of the «hollow» and the «screening» charged dust shells shows that their dynamics are somewhat similar. Also, note that for a family of concentric spherical charged shells, the generalized Hamiltonian constraint takes place. Full Hamiltonian (58) of the configuration numerically equals to the difference between active masses outside the system and inside it, i.e. $H = E_{tot} = M_N - M_0$.

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